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Two-Particle Spectrum of the Generator for Stochastic Model of Planar Rotators at High Temperatures

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We study two-particle spectrum branches of the generator in the stochastic model of planar rotators, using the construction of a special basis in two-particle invariant subspaces. We prove that the branches of the spectrum are in a small neighborhood of the point 2. We prove the existence of two bound states in addition to the continuous part of the spectrum in the one-dimensional case.

KEY WORDS: Langevin dynamics; invariant subspaces of the generator; spectral analysis of cluster operators.

1. INTRODUCTION AND THE MAIN RESULTS

The study of spectral properties of the generator in the stochastic model of planar rotators commenced with the paper of R. A. Minlos and Yu. G. Kondratiev.[1] The authors have suggested a new method for spectral analysis of such operators, similar in some sense to the cluster expansion method. In ref. 1 the authors constructed one-particle invariant subspaces of the generator and found the spectrum of the generator on these subspaces. This paper is a direct continuation of ref. 1: we investigate the next two-particle invariant subspaces of the generator and corresponding two-particle branches of the spectrum. In our study we use both the methods of ref. 1, and the spectral analysis of the cluster operators.[2]

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We define the generator H of the Langevin dynamics for stochastic model of planar rotators as a closure of the following differential operator

$$H^0 f(x) = -\sum_{k \in Z^d} \frac{\partial^2 f(x)}{\partial x_k^2} + \beta \sum_{k \in Z^d} \sum_{l:\,|k-l|=1} \frac{\partial f(x)}{\partial x_k} \sin(x_k - x_l)$$

$$D(H_\beta^0) = \{ f \in \mathscr{H} \mid f'' \in \mathscr{H} \}$$

where $\mathscr{H} = L_2(\Omega, d\mu_\beta)$, $\Omega = T^{Z^d}$ (T is the one-dimensional torus), $d\mu_\beta$ is the limit Gibbs measure on Ω in the planar rotators model with formal Hamiltonian

$$U(x) = -\sum_{|k-j|=1} \cos(x_k - x_j), \qquad x \in \Omega$$

β (inverse temperature) is small.

In ref. 3 it was proved that H is the selfadjoint operator in $\mathscr{H}$, and the stochastic semigroup

$$\mathscr{T}^t = e^{-tH}, \qquad t \geqslant 0$$

generates a reversible Markov process on Ω

$$x(t) = \{ x_k(t), k \in Z^d \}, \qquad x_k(t) \in T, \quad t \geqslant 0$$

with invariant measure μ_β. For any $f \in \mathscr{H}$:

$$(\mathscr{T}^t f)(x) = \langle f(x(t)) \rangle_{\mathscr{P}, x}$$

where $\langle \cdot \rangle_{\mathscr{P}, x}$ is a conditional average with respect to the distribution $\mathscr{P}$ of the process $x(t)$ with initial condition $x(0) = x$.

The operator H belongs to the family of so-called infinite-particle operators. Separation of one-particle, two-particle,... etc. branches of the spectrum is the powerful method for the spectral analysis of such operators (see ref. 2). We begin with the case $\beta = 0$. It is easy to see that the spectrum of the "non-perturbed" operator (with $\beta = 0$) consists of only eigenvalues 0, 1, 2,.... The eigenvalue 0 has multiplicity one, while the other eigenvalues 1, 2,... have infinite multiplicity. So we expect that when we "switch on the perturbation" ($\beta \neq 0$, β is small), the spectrum of H will be spreaded in small neighborhoods of the points 1, 2, 3,..., which had infinite multiplicity. Of course, the spectrum branches can overlap beginning with some number (depending on β).

The results of this paper supplement the similar ones from ref. 1, and together they give the explicit information about two lower branches of the spectrum: these branches are isolated (the one-particle branches are in a small neighborhood of the point 1, the two-particle ones are in a small neighborhood of the point 2) and separated from the rest of the spectrum. In addition the one-particle branches have only absolute continuous component, while the two-particle branches have also absolute continuous component and possibly a finite set of eigenvalues (discrete component).

The result relating to the leading one-particle branches of the spectrum is the following:[1]

Theorem 1 (Minlos, Kondratiev). For small enough β there exist two orthogonal subspaces $\mathscr{H}_1^{\pm} \subset \mathscr{H}$ which are invariant with respect to the operator H and the unitary group of the space translations U_j, $j \in Z^d$. The unitary involution $J: \mathscr{H} \to \mathscr{H}$

$$(Jf)(x) = f(-x)$$

transfers the subspaces $\mathscr{H}_1^{\pm}$ and the operators

$$H_1^{\pm} = H|_{\mathscr{H}_1^{\pm}}$$

each of them to other:

$$J\mathscr{H}_1^{\pm} = \mathscr{H}_1^{\mp}, \qquad \mathscr{J} H_1^{\pm} = H_1^{\mp} \mathscr{J}$$

The spectra of the operators $H_1^{\pm}$ and $U_j^{\pm} = U_j|_{\mathscr{H}_1^{\pm}}$ are the same as the ranges of the functions

$$m(\lambda) = 1 - \beta \sum_{s=1}^{d} \cos \lambda_s + \kappa(\beta, \lambda) \qquad \text{and} \qquad \exp\{i(\lambda, j)\}, \qquad \lambda \in T^d \qquad (1)$$

respectively. The function $m(\lambda)$ is an analytic function in a complex neighborhood W of the torus T^d and

$$|\kappa(\beta, \lambda)| \leqslant C_1 \beta^2, \qquad \lambda \in W$$

The spectrum of H in the orthogonal complement

$$\mathscr{H}^{\perp} = \mathscr{H} \ominus (\{\mathbf{1}\} \oplus \mathscr{H}_1^{+} \oplus \mathscr{H}_1^{-})$$

meets the following estimate:

$$\sigma(H_\beta|_{\mathscr{H}^{\perp}}) > 2 - C_2 \beta$$

where C_1, C_2 are constants.

We study here two-particle invariant subspaces of the operator H. To do this we are constructing below some appropriate basis. The idea of such basis was proposed by R. A. Minlos in the paper.[4] The operator H in this basis is appears to be similar to a cluster operator. Hence we can use the methods for the spectral analysis of the cluster operators to obtain the information about next two-particle branches of the spectrum for the operator H.

The main result is the following.

Theorem 2. For small enough β there exist three orthogonal subspaces $\mathscr{H}_2^+, \mathscr{H}_2^-, \mathscr{H}_2^0 \subset \mathscr{H}$ which are invariant with respect to the operator H and the unitary group of the space translations $\{U_j, j \in Z^d\}$. The unitary involution $(Jf)(x) = f(-x)$ transfers the subspaces $\mathscr{H}_2^\pm$ and the operators

$$H_2^\pm = H|_{\mathscr{H}_2^\pm}$$

each of them to other: $J\mathscr{H}_2^\pm = \mathscr{H}_2^\mp$, $JH_2^\pm = H_2^\mp \mathscr{J}$, so the spectra of H_2^+ and H_2^- are the same.

The spectra of the operators $H_2^\pm$ and $H_2^0 = H|_{\mathscr{H}_2^0}$ have an absolute continuous component (the Lebesgue branch of the spectrum) and possibly a finite number of eigenvalues ("energy" of corresponding bound states). The singular spectrum is empty.

The Lebesgue parts of the spectra for the operators $H_2^\pm$ and H_2^0 are the same as the range of the function

$$m(\lambda_1, \lambda_2) = m(\lambda_1) + m(\lambda_2)$$

where $m(\lambda)$ is the function defined in the Theorem 1.

In the one-dimensional case ($d = 1$) each of the operators $H_2^\pm$ and H_2^0 has an unique bound state with different eigenvalues ("energy").

The spectrum of H in the orthogonal complement

$$\mathscr{H}' = \mathscr{H} \ominus (\{\mathbf{1}\} \oplus \mathscr{H}_1 \oplus \mathscr{H}_2)$$

meets the estimate:

$$\sigma(H|_{\mathscr{H}'}) > 3 - C\beta$$

where

$$\mathscr{H}_1 = \mathscr{H}_1^+ \oplus \mathscr{H}_1^-, \qquad \mathscr{H}_2 = \mathscr{H}_2^+ \oplus \mathscr{H}_2^- \oplus \mathscr{H}_2^0$$

and C is a constant.

From the precise information about the two-particle branches of the spectrum and the results relative to the asymptotics of correlations decrease from ref. 9, one can obtain the asymptotics of the correlations

$$\langle f(x(0)), f(x(t))\rangle_{\mathscr{P}} = \langle f(x(0))\cdot f(x(t))\rangle_{\mathscr{P}} - \langle f(x(0))\rangle_{\mathscr{P}}\cdot\langle f(x(t))\rangle_{\mathscr{P}}$$
$$= (f(x), \mathscr{T}^t f(x))_{\mathscr{H}} - \langle f(x)\rangle^2_{\mu_\beta}$$

as $t\to\infty$, and functions f are from some family $\Psi\subset\mathscr{H}$. Here $\mathscr{P}$ is defined above distribution of the process $x(t)$, μ_β is the invariant measure of the process, $(\cdot,\cdot)_{\mathscr{H}}$ is the scalar product in the Hilbert space $\mathscr{H}$. We introduce a family $\Psi\subset\mathscr{H}$ of smooth functions f on an arbitrary finite set of variables $x_{a_1},\ldots, a_{a_m}$ ($\{a_1,\ldots, a_m\}\subset Z^d$), such that the Fourier decomposition for f does not contain the exponents with charge ± 1 (see the definition of the charge below, (5)), but contains at least one of the exponents:

$$\exp\{\pm ix_a \pm ix_b\}, \qquad a\neq b$$

For instance, $f = e^{ix_a + ix_b}\in\Psi$ for any $a\neq b$. As will be seen from the description of the invariant subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$, any function $f\in\Psi$ has zero projection on the invariant subspace $\mathscr{H}_1 = \mathscr{H}_1^+\oplus\mathscr{H}_1^-$ and non-zero projection on the invariant subspace $\mathscr{H}_2$. From this fact and also from the 3 reasoning and results of the paper[9] we obtain

Corollary. Let $f\in\Psi$. Then

$$\langle f(x(0)), f(x(t))\rangle_{\mathscr{P}} = \begin{cases} \dfrac{K_1(f)}{t^2}\, e^{-2tm_0}(1+o(1)), & d=1\\[2mm] \dfrac{K_2(f)}{t^2(\ln t)^2}\, e^{-2tm_0}(1+o(1)), & d=2\\[2mm] \dfrac{K_d(f)}{t^d}\, e^{-2tm_0}(1+o(1)), & d\geqslant 3\end{cases}$$

as $t\to\infty$. Here $K_d(f)$ are constants depending on the function f and the dimension d, m_0 is the minimum of the function $m(\lambda)$ (1).

2. THE CONSTRUCTION OF THE TWO-PARTICLE INVARIANT SUBSPACE

Let $\mathscr{M}$ be the set of all multi-indices (integer-value functions with bounded support):

$$\mathscr{M} = \{n = (n(k))_{k\in Z^d}\}, \qquad n(k)\in Z, \quad n(k) = 0 \qquad \text{when} \quad |k| > N(n)$$

Then the set of functions $L_{\mathcal{M}}$

$$L_{\mathcal{M}} = \left\{ e_n(x) = \prod_{k \in \operatorname{supp} n} \exp\{in(k)\, x_k\},\ n \in \mathcal{M} \right\}$$

forms the basis (non-orthogonal) in $\mathcal{H} = L_2(\Omega, d\mu_\beta)$.

We denote by $L \subset \mathcal{H}$ the space of functions of the form:

$$f(x) = \sum_{n \in \mathcal{M}} f_n e_n(x) \tag{2}$$

with

$$\|f\|_L \equiv \sum_{n \in \mathcal{M}} |f_n| < \infty \tag{3}$$

Under condition (3) the series (2) converges absolutely and uniformly. In addition the space L is dense in $\mathcal{H}$, and for any $f \in L$:

$$\|f\|_{\mathcal{H}} \leqslant \|f\|_L$$

Let $B_{n,n'}$ be matrix elements of a bounded operator B in L with respect to the basis $\{e_n(x)\}$:

$$Be_n = \sum_{n' \in \mathcal{M}} B_{n,n'}\, e_{n'}$$

Then the norm of the operator B in the space L is defined as

$$\|B\|_L = \sup_n \sum_{n' \in \mathcal{M}} |B_{n,n'}|$$

Further we need the following useful lemma.

Lemma 1.[5] Let B be a symmetric operator in $\mathcal{H}$ such that $B: L \to L$, and the restriction $B|_L$ is a bounded operator in L. Then B is a bounded operator in $\mathcal{H}$ with

$$\|B\|_{\mathcal{H}} \leqslant \|B|_L\|_L \tag{4}$$

The proof see in ref. 5.

For any function $n \in \mathcal{M}$ we introduce two notations. Let

$$r(n) = \sum_{k \in \operatorname{supp} n} n(k) \tag{5}$$

be a charge of n, and

$$|n| = \sum_{k \in \operatorname{supp} n} |n(k)|$$

be a modulus of n.

Since the stochastic dynamics holds the charge, then the space $\mathscr{H}$ (and L) is decomposed into the direct sum of subspaces of the fixed charge which are invariant with respect to H:

$$\mathscr{H} = \bigoplus_{r \in Z} \overline{\mathscr{L}_r}, \qquad L = \bigoplus_{r \in Z} \mathscr{L}_r \tag{6}$$

where $\mathscr{L}_r$ is the linear span of the vectors $\{e_n(x) : r(n) = r\}$, $r \in Z$, and we consider a closure $\overline{\mathscr{L}_r}$ in a sense of the norm in the space $\mathscr{H}$. In particular, we have

$$\mathscr{H}_1 \subset (\overline{\mathscr{L}_1} \oplus \overline{\mathscr{L}_{-1}}), \qquad \mathscr{H}_1^{\pm} \subset \overline{\mathscr{L}_{\pm 1}}$$

and

$$\mathscr{H}_2 \subset (\overline{\mathscr{L}_2} \oplus \overline{\mathscr{L}_{-2}} \oplus \overline{\mathscr{L}_0} \ominus \{\mathbf{1}\})$$

The decomposition (6) implies that

$$\mathscr{H}_2 = \mathscr{H}_2^+ \oplus \mathscr{H}_2^- \oplus \mathscr{H}_2^0$$

with

$$\mathscr{H}_2^{\pm} \subset \overline{\mathscr{L}_{\pm 2}}, \qquad \mathscr{H}_2^0 \subset \overline{\mathscr{L}_0} \ominus \{\mathbf{1}\}$$

We need next some methods and constructions from the book[2] and the paper,[1] so we briefly remind them here.

According to the classical scheme of the construction of the invariant subspaces (see ref. 2), the subspace $\mathscr{H}_1^+$ (and $\mathscr{H}_1^-$ analogously) is the closure of the following subspace

$$\mathscr{N}_1^+ = L_1^+ + S^+ L_1^+ \tag{7}$$

Here $L_1^+ \subset \mathscr{L}_1$ is the linear span of the vectors $\{e^{ix_k}, k \in Z^d, \}$ and S^+ is a operator

$$S^+ : L_1^+ \to L_{>1}^+$$

where $L^+_{>1} = \mathscr{L}_1 \ominus L^+_1$ is the linear span of vectors of the form $\{e_n(x), r(n) = 1, |n| > 1\}$. The operator S^+ meets the equation

$$S^+ = -(H^{(1)}_{11})^{-1} H^{(1)}_{10} + (H^{(1)}_{11})^{-1} S^+ H^{(1)}_{00} + (H^{(1)}_{11})^{-1} S^+ H^{(1)}_{01} S^+ \quad (8)$$

Here the operators $H^{(1)}_{00} : L^+_1 \to L^+_1$, $H^{(1)}_{01} : L^+_{>1} \to L^+_1$ etc. correspond to the following matrix representation for the operator $H|_{\mathscr{L}_1}$ under the decomposition $\mathscr{L}_1 = L^+_1 \oplus L^+_{>1}$:

$$H|_{\mathscr{L}_1} = \begin{pmatrix} H^{(1)}_{00} & H^{(1)}_{01} \\ H^{(1)}_{10} & H^{(1)}_{11} \end{pmatrix} \quad (9)$$

In ref. 1 it was proved that the operator S^+ meets the bound

$$\|S^+\| < C\beta \quad (10)$$

with a constant C, where $\|\cdot\|$ is the norm in the space of bounded operators $\mathscr{L}(L^+_1 \to L^+_{>1})$.

Thus we have (nonorthogonal) basis vectors in the space $\mathscr{H}^+_1$ of the form

$$h_r = e_r + S^+ e_r, \qquad r \in Z^d \quad (11)$$

where $e_r = e^{ix_r}$. In our case $U_j h_r = h_{r+j}$, $r, j \in Z^d$.

To study two-particle subspaces we have to get more concrete information about the spectrum of the generator restricted on the one-particle subspaces $\mathscr{H}^\pm_1$. One can find this information in the lemmas 2–5.

Lemma 2. The spectrum of the operator $H^+_1 = H|_{\mathscr{H}^+_1}$ (and $H^-_1 = H|_{\mathscr{H}^-_1}$ is the same as the range of the function

$$m(\lambda) = A - B\beta \sum_{j=1}^{d} \cos \lambda_j + C\beta^4 \sum_{j,r=1}^{d} \cos(\lambda_j + \lambda_r) + m_1(\beta, \lambda) \quad (12)$$

where $A = 1 + O(\beta^2)$, $B = 1 + O(\beta^2)$, C are constants,

$$|m_1(\beta, \lambda)| \leqslant \text{const} \cdot \beta^5$$

for all $\lambda \in T^d$.

Proof. The lemma improves the results of the paper,[1] and we use here the technique of ref. 1 complemented with some additional estimates.

As noted above, the one-particle invariant subspace $\mathscr{H}_1^+$ has the form (7), and matrix representation (9) implies the following formula for the matrix elements of the operator H_1^+:

$$(H_1^+)_{r,s} = m(r-s) = (H_{00}^{(1)})_{r-s} + (H_{01}^{(1)} S^+)_{r-s} \tag{13}$$

We have used here the fact that the operator H_1^+ commutes with the group of space translations U_j, $j \in Z^d$, and hence H_1^+ is a convolution:

$$(H_1^+)_{r,s} = (H_1^+)_{r-s}$$

The definition of the operator $H_{00}^{(1)}$ (see (9)) implies that

$$(H_{00}^{(1)})_{r-s} = \begin{cases} 1, & r = s \\ -\dfrac{\beta}{2}, & |r-s| = 1 \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

and moreover $(H_{01}^{(1)} S^+)_{r-s} = O(\beta^2)$. We have to get now the representation for $(H_{01}^{(1)} S^+)_{r-s}$ up to the order $O(\beta^4)$. For this purpose we invoke a lemma from ref. 6.

Lemma 3. For small enough β the estimate for the matrix elements of the operator $H_{01}^{(1)} S^+$ is the following:

$$|(H_{01}^{(1)} S^+)_{r-s}| \leqslant d_{r-s} \beta^2 (C\beta)^{|r-s|} \tag{15}$$

with

$$\sum_{u \in Z^d} |d_u| < D$$

and C, D are constants.

Proof see in ref. 6.

Then (13)–(15) imply that the matrix elements $m(r-s)$ of the operator H_1^+ have the bound:

$$|m(r-s)| \leqslant R_{r-s} (C\beta)^{|r-s|} \tag{16}$$

where $\sum_u |R_u| < 3/2 + D\beta^2$, C, D are constants.

We also use the following lemma from the paper.[1]

Lemma 4. Let $\hat{m}(r-s)$ be matrix elements of H_1^+ in the orthonormal basis $\{\hat{h}_r, r \in Z^d\}$ obtained with help of Gramm matrix from the basis $\{h_r, r \in Z^d\}$ (11):

$$H_1^+ \hat{h}_r = \sum_s \hat{m}(r-s)\, \hat{h}_s \tag{17}$$

Then $\hat{m}(r-s) = m(r-s)$, where the function $m(r-s)$ is defined by the formula (13).

Proof see in ref. 1.

Now from (13)–(15), (17) after Fourier transform

$$\hat{h}_r \to e^{i(\lambda, r)}, \qquad \lambda \in T^d$$

we obtain the representation (12). Lemma 2 is proved.

In what follows we shall need some sharp estimates on the matrix elements of the operator S^+.

Lemma 5. For any $k \in Z^d$ and $n \in \mathcal{M}$ with $r(n) = 1$, $|n| > 1$ we have

$$|S_{k,n}^+| = D_{k,n}(C\sqrt{\beta})^{(1/2)\, d_{\{k, \operatorname{supp} n\}} + (1/4)\,|n|} \tag{18}$$

where

$$\sup_k \sum_{n \in \mathcal{M}} |D_{k,n}| < D\beta^{3/8}$$

C, D are constants, $d_{\{k, \operatorname{supp} n\}}$ is the length of a minimal connected set of bonds of the lattice Z^d containing the point k and all points of $\operatorname{supp} n$.

Proof. Lemma 5 is a sharpening of the analogous statement from ref. 1. To prove (18) we have to consider the space of operators $\mathscr{A} = \{Q : L_1^+ \to L_{>1}^+\}$ with the norm

$$|||Q||| = \sup_k \sum_n |Q_{k,n}| \left(\frac{1}{C\sqrt{\beta}}\right)^{(1/2)\, d_{\{k, \operatorname{supp} n\}} + (1/4)\,|n|}$$

The estimates on the norms of all operator from the equation (8) can be obtained by analogy with the reasoning of ref. 1:

$$|||(H_{11}^{(1)})^{-1} S^+ H_{00}^{(1)}||| \leqslant \tfrac{1}{2}(1+k_1\beta)\, |||S^+|||$$

$$|||(H_{11}^{(1)})^{-1} S^+ H_{01}^{(1)} S^+||| \leqslant k_2\beta)\, |||S^+|||^2$$

$$|||(H_{11}^{(1)})^{-1} H_{10}^{(1)}||| \leqslant k_3\beta^{3/8}$$

Here $k_j = k_j(d), j = 1, 2, 3$ are constants depending only on the dimension d.

Then by applying the contraction maps principle to the equation (8) from above estimates it follows that $S^+ \in \mathscr{A}$ and $|||S^+||| < D\beta^{3/8}$, D is a constant. Lemma is proved.

We now pass to the proof of the Theorem 2. The first step is to construct two-particle invariant subspace $\mathscr{H}_2^+$. Our reasoning is similar to the previous one, when we have isolated the one-particle invariant subspaces. We denote by L_2 the linear span of products $\{h_r \cdot h_s, r \neq s\}$ of the basis vectors $h_r, h_s, r \neq s$, from the one-particle invariant subspace $\mathscr{H}_1^+$, and by $L_{>2}$ the subspace

$$L_{>2} = \mathscr{L}_2 \ominus L_2 \tag{19}$$

We consider a basis in $L_{>2}$ of the following form:

$$\{h_r \cdot h_r \text{ and } e_n(x), \text{ with } r(n) = 2, |n| > 2\}$$

It is clear from estimate (10) that the operator $G: \mathscr{L}_2 \to \mathscr{L}_2$, setting the transformation of the classical basis in $\mathscr{L}_2$ of the form $\{e_n(x), r(n) = 2\}$ to the basis

$$\{u_n(x) = h_r \cdot h_s, \text{ when } |n| = 2; u_n(x) = e_n(x), \text{ when } |n| > 2\}$$

has the form

$$G = E + \delta \tag{20}$$

where E is the identity matrix, and $\|\delta\| \leqslant C\beta$.

The decomposition (19) generates the following matrix representation for the operator $H_2 = H|_{\mathscr{L}_2}$:

$$H_2 = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix} \tag{21}$$

where $H_{00}: L_2 \to L_2$, $H_{01}: L_{>2} \to L_2$ etc. We shall find the invariant subspace $\mathscr{H}_2^+ \subset \bar{\mathscr{L}}_2$ as the closure of the following subspace

$$\mathscr{N}_2^+ = L_2 + S_2 L_2 \tag{22}$$

where $S_2: L_2 \to L_{>2}$.

Remark. In a similar way we can construct the invariant subspaces $\mathscr{H}_2^- \subset \bar{\mathscr{L}}_{-2}$ and $\mathscr{H}_2^0 \subset \bar{\mathscr{L}}_0 \ominus \{\mathbf{1}\}$. The first one is a perturbation by the operator S_2^- of the linear span of the vectors $\{h_r^{(-)} \cdot h_s^{(-)}, r \neq s\}$, where $h_r^{(-)}$, $h_s^{(-)}$ are the basis vectors of $\mathscr{H}_1^-$, and the second one is the analogous perturbation of the linear span of the vectors $\{h_r^{(-)} \cdot h_s, r \neq s\}$.

The decomposition (21) implies that the existence of the invariant subspace $\mathscr{H}_2^+$ of the form (22) is equivalent to the existence of a solution for the equation on S_2:

$$H_{11} S_2 + H_{10} = S_2 H_{00} + S_2 H_{01} S_2 \tag{23}$$

The result is the following.

Lemma 6. For small enough β there exists a unique solution S_2 of the equation (23) with

$$\|S_2\| \leqslant C\beta \tag{24}$$

where $\|\cdot\|$ is the norm in the space $\mathscr{L}(L_2 \to L_{>2})$ of linear bounded operators, C is a constant.

We denote by

$$\mathscr{H}' = \mathscr{H} \ominus (\{\mathbf{1}\} \oplus \mathscr{H}_1 \oplus \mathscr{H}_2)$$

where $\mathscr{H}_2 = \mathscr{H}_2^+ \oplus \mathscr{H}_2^- \oplus \mathscr{H}_2^0$ is the two-particle invariant subspace.

Lemma 7. The spectrum of the operator $H|_{\mathscr{H}'}$ has the following lower bound:

$$\sigma(H|_{\mathscr{H}'}) > 3 - C\beta$$

where C is a constant.

Using (20) and (4), Lemmas 6 and 7 can be proved by analogy with the proof of the similar results from ref. 1 relative to the one-particle invariant subspaces.

3. SPECTRAL ANALYSIS OF THE GENERATOR *H* ON THE TWO-PARTICLE INVARIANT SUBSPACES

We proceed now to the spectral analysis of the operator H restricted on the two-particle invariant subspaces, and first we formulate relevant lemmas.

Lemma 8. If $r \neq s$, then

$$H(h_r \cdot h_s) = (H_1^+ h_r)\, h_s + (H_1^+ h_s)\, h_r + \Delta^{r,s}$$
$$= \sum_{r' \in Z^d,\, r' \neq s} m(r-r')\, h_{r'} h_s + \sum_{s' \in Z^d,\, s' \neq r} m(s-s')\, h_r h_{s'} + \tilde{\Delta}^{r,s} \qquad (25)$$

Here

$$\Delta^{r,s}(x) = \sum_{w \in \mathscr{M}} d_w^{r,s} e_w(x) \in \mathscr{L}_2$$
$$\tilde{\Delta}^{r,s} = \Delta^{r,s} + m(r-s)(h_r h_r + h_s h_s)$$

and

$$|d_w^{r,s}| < D(\alpha_2 \sqrt{\beta})^{(1/2)\, d_{\{r,s,\operatorname{supp} w\}} + (1/4)\,|w|} \qquad (26)$$

where $d_{\{r,s,\operatorname{supp} w\}}$ is the length of a minimal connected subgraph containing the points r, s and all points of the supp w; α_2, D are constants.

Proof of Lemma 8 see in Appendix.

Corollary. Under the decompositions (19) and (21) we have

$$\Delta^{r,s}(x) = g_1^{r,s}(x) + g_2^{r,s}(x)$$
$$\tilde{\Delta}^{r,s}(x) = g_1^{r,s}(x) + \tilde{g}_2^{r,s}(x) = g_1^{r,s}(x) + (g_2^{r,s}(x) + m(r-s)(h_r h_r + h_s h_s)) \qquad (27)$$

with

$$g_1^{r,s}(x) \in L_2, \qquad \tilde{g}_2^{r,s}(x) \equiv H_{10}(h_r \cdot h_s) \in L_{>2}$$

and

$$|(g_1^{r,s})_{r',s'}| \leqslant D(\alpha_2 \sqrt{\beta})^{(1/2)\, d_{\{r,s,r',s'\}} + (1/2)} \qquad (28)$$
$$|(\tilde{g}_2^{r,s})_w| \leqslant D(\alpha_2 \sqrt{\beta})^{(1/2)\, d_{\{r,s,\operatorname{supp} w\}} + (1/4)\,|w|} \qquad (29)$$

Here α_2, D are the same constants as in the Lemma 8.

Lemma 9. The matrix elements of the operator $S_2: L_2 \to L_{>2}$ have the bound:

$$|(S_2)_w^{r,s}| \leqslant C_w^{r,s}(\tilde{\alpha}\sqrt{\beta})^{(1/2)\, d_{\{r,s,\operatorname{supp} w\}} + (1/4)\,|w|} \tag{30}$$

where

$$\sup_{r \neq s} \sum_w |C_w^{r,s}| < C$$

C, $\tilde{\alpha} > \alpha_2$ are constants.

Proof of Lemma 9 see in Appendix.

Let us consider now the generator H on the two-particle invariant subspace. Lemma 6 implies that a (non-orthogonal) basis in the subspace $\mathscr{H}_2^+$ has the following form:

$$b_{r,s} = h_r \cdot h_s + S_2(h_r \cdot h_s), \qquad r \neq s \tag{31}$$

and the operator H_2^+ is written in this basis as:

$$H_2^+ b_{r,s} = \sum_{r' \neq s} m(r - r')\, b_{r',s} + \sum_{s' \neq r} m(s - s')\, b_{r,s'} + \sum_{r', s', r' \neq s'} K(r, s; r', s')\, b_{r',s'} \tag{32}$$

where $m(r - r')$ is the same function as in the formula (13). The representation (21) implies that the matrix elements of the operator H_2^+ in the basis $\{b_{r,s}\}$ (31) have the form:

$$(H_2^+)_{(r,s),(r',s')} = (H_{00})_{(r,s),(r',s')} + (H_{01} S_2)_{(r,s),(r',s')}$$

where

$$(H_{00})_{(r,s),(r',s')},\ (H_{01} S_2)_{(r,s),(r',s')}$$

are the matrix elements of the operators H_{00} and $H_{01} S_2$ in the basis $\{h_r \cdot h_s\}$ respectively. Then from the representations (25), (27) and (32) we get the formula for the kernel of the operator H_2^+:

$$K(r, s; r', s') = (g_1^{r,s})_{(r',s')} + (H_{01} S_2)_{(r,s),(r',s')} \tag{33}$$

and hence from the estimates (28), (29) and (30) we have

$$|K(r, s; r', s')| \leqslant D(\tilde{\alpha}\sqrt{\beta})^{(1/2)\, d_{\{r,s,r',s'\}} + (1/2)} \tag{34}$$

with constants D and $\tilde{\alpha}$.

Let us recall that we have considered above the case when $h_r, h_s \in \mathscr{H}_1^+$, and $\mathscr{H}_2^+ \subset \mathscr{L}_2$. In a similar manner we can construct the invariant subspaces $\mathscr{H}_2^- \subset \mathscr{L}_{-2}$ and $\mathscr{H}_2^0 \subset \mathscr{L}_0 \ominus \{1\}$.

Since the unitary involution $(Jf)(x) = f(-x)$ transfers the subspaces $\mathscr{H}_2^\pm$ each of them to other: $J\mathscr{H}_2^\pm = \mathscr{H}_2^\mp$, then the spectra of the operators H_2^+ and H_2^- are the same. In addition Theorem 1 and formula (32) imply that the absolute continuous spectra of the operators $H_2^\pm$ and H_2^0 are coinciding, and they are the same as the range of the function

$$m(\lambda_1, \lambda_2) = m(\lambda_1) + m(\lambda_2), \qquad \lambda_1, \lambda_2 \in T^d$$

For the further study of the spectrum of the operators $H_2^\pm$ and H_2^0 in the one-dimensional case we pass to the Fourier transform:

$$Q: b_{r,s} \to \exp\{ir\lambda_1 + is\lambda_2\}, \qquad r \neq s, \quad \lambda_1, \lambda_2 \in T$$

$$Q: \mathscr{H}_2^\pm(\mathscr{H}_2^0) \to \tilde{L}_2^{sym}(T \times T) \subset L_2^{sym}(T \times T)$$

where $\tilde{L}_2^{sym}(T \times T)$ is the Hilbert space of symmetric functions $f(\lambda_1, \lambda_2) = f(\lambda_2, \lambda_1)$, $\lambda_1, \lambda_2 \in T$, such that

$$\int_{T \times T} f(\lambda_1, \lambda_2)\, h(\lambda_1 + \lambda_2)\, d\lambda_1\, d\lambda_2 = 0$$

for any $h(\lambda) \in L_2(T)$.

The transformation Q is not orthogonal, it can be represented as a composition of two transformations:

$$Q = FG^{-1/2}$$

where G is the Gramm matrix for the basis $\{b_{r,s}\}$, and $G^{-1/2}$ transforms the basis $\{b_{r,s}\}$ to the orthonormal basis $\{\hat{b}_{r,s}\}$, so

$$F: \hat{b}_{r,s} \to \exp\{ir\lambda_1 + is\lambda_2\}$$

is the orthogonal transformation. From (31), (11), (10), (24), (30) it is easy to show that the Gramm matrix has the form

$$G = E + \Delta \tag{35}$$

where E is the identity matrix, $\|\Delta\| < C\beta$, with a constant C. The representation (35) implies the existence of the transformations $G^{-1/2}$, $G^{1/2}$. Hence the transformation $Q = FG^{-1/2}$ is reversible and the operators H_2^+ and $\tilde{H}_2^+ = QH_2^+ Q^{-1} : \tilde{L}_2^{sym} \to \tilde{L}_2^{sym}$ are similar. We use here a lemma from the book.[7]

Lemma 10. The spectra of similar operators coincide.

Proof see in ref. 7.

Now we proceed to study the spectra of the operators $\tilde{H}_2^+$ and $\tilde{H}_2^0$. The results of ref. 8 imply that the operator $\tilde{H}_2^+$ has the form

$$\begin{aligned}(\tilde{H}_2^+ f)(\lambda_1, \lambda_2) &= m(\lambda_1, \lambda_2)\, f(\lambda_1, \lambda_2) + \int_{T \times T} \tilde{K}(\lambda_1, \lambda_2, \mu_1, \mu_2) \\ &\quad \times \delta(\lambda_1 + \lambda_2 - \mu_1 - \mu_2)\, f(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 \end{aligned} \tag{36}$$

Here

$$\begin{aligned} m(\lambda_1, \lambda_2) = m(\lambda_1) + m(\lambda_2) &= 2A - B\beta(\cos \lambda_1 + \cos \lambda_2) \\ &\quad + C\beta^4(\cos 2\lambda_1 + \cos 2\lambda_2) + O(\beta^5) \end{aligned} \tag{37}$$

$A = 1 + O(\beta^2)$, $B = 1 + O(\beta^2)$, the function $m(\lambda)$ is defined in Lemma 1,

$$\begin{aligned} \tilde{K}(\lambda_1, \lambda_2, \mu_1, \mu_2) &= -m(\lambda_1, \lambda_2) - m(\mu_1, \mu_2) + \int_{\xi_1 + \xi_2 = \lambda_1 + \lambda_2} m(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 \\ &\quad + \tilde{K}_1(\lambda_1, \lambda_1, \mu_1, \mu_2) \end{aligned} \tag{38}$$

where $\tilde{K}_1(\lambda_1, \lambda_2, \mu_1, \mu_2)$ is the Fourier transform of the function $K(r, s, r', s')$; $r \neq s$, $r' \neq s'$. The operator $\tilde{H}_2^0$ has the analogous representation, and let $\tilde{K}^0(\lambda_1, \lambda_2, \mu_1, \mu_2)$ be the kernel of its integral part:

$$\begin{aligned} \tilde{K}^0(\lambda_1, \lambda_2, \mu_1, \mu_2) &= -m(\lambda_1, \lambda_2) - m(\mu_1, \mu_2) + \int_{\xi_1 + \xi_2 = \lambda_1 + \lambda_2} m(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 \\ &\quad + \tilde{K}_1^0(\lambda_1, \lambda_2, \mu_1, \mu_2) \end{aligned}$$

where $\tilde{K}_1^0(\lambda_1, \lambda_2, \mu_1, \mu_2)$ is the Fourier transform of the function $K^0(r, s, r', s')$; $r \neq s$, $r' \neq s'$.

Remark. From the estimates (34) it follows that the functions $\tilde{K}(\Lambda, \lambda, \mu)$ and $\tilde{K}^0(\Lambda, \lambda, \mu)$, where $\Lambda = \lambda_1 + \lambda_2 = \mu_1 + \mu_2$, $\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)$, $\mu = \frac{1}{2}(\mu_1 - \mu_2)$ are analytic with respect to $\Lambda, \lambda, \mu \in T$.

Lemma 11. In the one-dimensional case for small enough β there exists a neighborhood $O_\beta(\pi)$ of the point π (which has order β) such that for any $\Lambda = \lambda_1 + \lambda_2 \in O_\beta(\pi)$ the operators $\tilde{H}_2^+$ and $\tilde{H}_2^0$ have the unique eigen-states with eigenvalue:

$$\omega(\Lambda) = 2A - C_1(\Lambda)\,\beta^2 + O(\beta^3), \qquad C_1(\Lambda) \geqslant 0$$

and respectively

$$\omega^0(\Lambda) = 2A + C_0(\Lambda)\,\beta^2 + O(\beta^3), \qquad C_0(\Lambda) \geqslant 0$$

The operators $\tilde{H}_2^+$ and $\tilde{H}_2^0$ have no eigen-states when $\Lambda \notin O_\beta(\pi)$.

Here $A = 1 + O(\beta^2)$ is the same constant as in the representation (37) for the function $m(\lambda_1, \lambda_2)$.

Proof. Let us consider the operator $\tilde{H}_2^+$ (the operator $\tilde{H}_2^0$ can be investigated by a similar way). To study a discrete spectrum of the operators $\tilde{H}_2^+$ and $\tilde{H}_2^0$ we need both the estimate (34) and a detailed information about the structure of the kernels $K(r, s, r', s')$ and $K^0(r, s, r', s')$. We note that the functions $K(r, s, r', s')$ and $K^0(r, s, r', s')$ are symmetric with respect to any pair of the variables (r, s) and (r', s') separately:

$$K(r, s, r', s') = K(s, r, r', s') = K(r, s, s', r') = K(s, r, s', r')$$

Hence in what follows we can assume that $r < s$.

Proposition. 1. If $|r - s| = 1$ then

$$K(r, s, r', s') = \begin{cases} -3\beta^2/8, & \text{when} \quad (r', s') = (r, s) \\ \beta^2/8, & \text{when} \quad (r', s') = (r-1, r) \quad \text{or} \quad (s, s+1) \end{cases} \tag{39}$$

$$K^0(r, s, r', s') = \begin{cases} -7\beta^2/24, & \text{when} \quad (r', s') = (r, s) \\ -5\beta^2/16, & \text{when} \quad (r', s') = (r-1, r) \quad \text{or} \quad (s, s+1) \end{cases} \tag{40}$$

In the other cases we have

$$|K(r, s, r', s')| \leqslant C_1 \beta^3, \qquad |K^0(r, s, r', s')| \leqslant C_1 \beta^3 \tag{41}$$

2. If $|r-s| \geqslant 2$ then for any r', s' the following estimates hold:

$$|K(r, s, r', s')| \leqslant C_2 \beta^3, \qquad |K^0(r, s, r', s')| \leqslant C_2 \beta^3 \tag{42}$$

Here C_1, C_2 are constants.

Proof see in Appendix.

We consider now the operator $\tilde{H}_2^+$ with respect to new variables $\Lambda = \lambda_1 + \lambda_2 = \mu_1 + \mu_2$, $\lambda = \frac{1}{2}(\lambda_1 - \lambda_2)$. The representation (36) yields that the operator $\tilde{H}_2^+$ could be written as:

$$(\tilde{H}_2^+ f)(\Lambda, \lambda) = m(\Lambda, \lambda)\, f(\Lambda, \lambda) + \int_T \tilde{K}_\Lambda(\lambda, \mu)\, f(\Lambda, \mu)\, d\mu$$

Hence $\tilde{H}_2^+$ is a direct integral (with respect to Λ) of the family of operators

$$(\tilde{H}_\Lambda f_\Lambda)(\lambda) = m_\Lambda(\lambda)\, f_\Lambda(\lambda) + \int_T \tilde{K}_\Lambda(\lambda, \mu)\, f_\Lambda(\mu)\, d\mu$$

acting for every fixed $\Lambda \in T$ in the spaces $\tilde{L}_2^{ev} \subset L_2^{ev}$ of even function $f_\Lambda(\lambda) \in L_2^{ev}(T)$ such that $\int_T f_\Lambda(\lambda)\, d\lambda = 0$. In doing so the functions $m_\Lambda(\lambda)$ and $\tilde{K}_\Lambda(\lambda, \mu)$ have the form:

$$m_\Lambda(\lambda) = 2A - 2\beta B \cos\frac{\Lambda}{2} \cos\lambda + 2\beta^4 C \cos\Lambda \cos 2\lambda + O(\beta^5)$$

$$\tilde{K}_\Lambda(\lambda, \mu) = -m_\Lambda(\lambda) - m_\Lambda(\mu) + \int_T m_\Lambda(\xi)\, d\xi + \tilde{K}_1(\Lambda, \lambda, \mu)$$

under notations of formulas (37) and (38).

We note that critical points of the function $m_\Lambda(\lambda)$ are non-degenerate for any Λ, and in addition all critical values of $m_\Lambda(\lambda)$ are simple when $\Lambda \neq \pi$. Then from results of the paper[8] it follows that the operator $\tilde{H}_\Lambda$ has no bound states for small enough β and for all $\Lambda \notin O(\pi)$. Here $O(\pi)$ is a small neighborhood of point π.

Let us consider the case $\Lambda = \pi$. We have:

$$m_\pi(\lambda) = 2A - 2\beta^4 C \cos 2\lambda + O(\beta^6)$$

$$\tilde{K}_\pi(\lambda, \mu) = -m_\pi(\lambda) - m_\pi(\mu) + \int_T m_\pi(\xi)\, d\xi - 4\beta^2 \cos\lambda \cos\mu + k_\pi(\beta, \lambda, \mu) \tag{43}$$

where $|k_\pi(\beta, \lambda, \mu)| \leqslant k\beta^3$ for any $\lambda, \mu \in T$; C, k are constants. This representation is the direct consequence of the proposition.

We note that

$$\tilde{H}_A : \tilde{L}_2^{ev} \to \tilde{L}_2^{ev}, \qquad \text{and} \qquad \tilde{H}_A : \{\text{const}\} \to 0 \tag{44}$$

From (44) it follows that if $f_A \in L_2^{ev}$ is an eigen-function of the operator $\tilde{H}_A$ with eigenvalue $w \neq 0$, then this function $f_A \in \tilde{L}_2^{ev}$, i.e., $\int_T f_A(\lambda)\, d\lambda = 0$. From (44) it also follows that the constant in the kernel $\tilde{K}_A(\lambda, \mu)$ does not vary the action of the operator $\tilde{H}_A$. Hence by (43) we can rewrite the function $\tilde{K}_\pi(\lambda, \mu)$ as:

$$\tilde{K}_\pi(\lambda, \mu) = -4\beta^2 \cos \lambda \cos \mu + \varphi_\pi(\lambda, \mu) \tag{45}$$

where $|\varphi_\pi(\lambda, \mu)| \leqslant C\beta^3$, C is a constant.

We consider further the operator

$$(\tilde{H}_\pi^{(b)} f)(\lambda) = m_\pi(\lambda)\, f(\lambda) - 4\beta^2 \cos \lambda \int_T \cos \mu f(\mu)\, d\mu$$

with the kernel

$$\tilde{K}_\pi^{(b)}(\lambda, \mu) = -4\beta^2 \cos \lambda \cos \mu$$

It is easy to see that the operator $\tilde{H}_\pi^{(b)}$ has the unique point of the discrete spectrum:

$$w_\pi^{(b)} = 2A - 2\beta^2 + u_0(\beta), \qquad |u_0(\beta)| \leqslant C_0 \beta^3$$

which is outside of the continuous spectrum of $\tilde{H}_\pi^{(b)}$ (or $\tilde{H}_\pi$):

$$\sigma_{ac}(\tilde{H}_\pi^{(b)}) = \sigma_{ac}(\tilde{H}_\pi) = \operatorname{Im} m_\pi(\lambda) \subset (2A - C_1\beta^4, 2A + C_1\beta^4)$$

C_1 is a constant.

Then (45) and the general perturbation theory imply that the operator $\tilde{H}_\pi$ has the unique point of the discrete spectrum w_π which is in a β^3-neighborhood of the point $w_\pi^{(b)}$. Finally, from the analyticity of the functions $m_A(\lambda)$ and $\tilde{K}_A(\lambda, \mu)$ with respect to A we obtain the statement of Lemma 11 for all $A \in O_\beta(\pi)$.

Lemma 11 is proved.

Theorem 2 is proved completely.

APPENDIX

A.1. The Proof of Lemma 8

For simplicity we consider the one-dimensional case: $d=1$. Let $f(x)$, $g(x)$ be functions from $\mathscr{H}$, where $x=\{x_k \in T, k \in Z^1\}$ is a configuration of the field. Then

$$H(f(x)\cdot g(x))$$
$$= -\sum_{k\in Z}\frac{\partial^2}{\partial x_k^2}(f(x)\cdot g(x))+\beta\sum_{k\in Z}b_k(x)\frac{\partial}{\partial x_k}(f(x)\cdot g(x))$$
$$=\left\{-\sum_{k\in Z}\frac{\partial^2}{\partial x_k^2}f(x)+\beta\sum_{k\in Z}b_k(x)\frac{\partial f(x)}{\partial x_k}\right\}\cdot g(x)$$
$$+\left\{-\sum_{k\in Z}\frac{\partial^2}{\partial x_k^2}g(x)+\beta\sum_{k\in Z}b_k(x)\frac{\partial g(x)}{\partial x_k}\right\}\cdot f(x)-2\sum_{k\in Z}\frac{\partial f(x)}{\partial x_k}\cdot\frac{\partial g(x)}{\partial x_k}$$

with

$$b_k(x)=\sum_{l\in Z,\ |k-l|=1}\sin(x_k-x_l)$$

We put $f(x)=h_r(x)$, $g(x)=h_s(x)$, then

$$\Delta^{r,s}=-2\sum_{k\in Z}\frac{\partial h_r}{\partial x_k}\cdot\frac{\partial h_s}{\partial x_k} \tag{46}$$

To bound $\Delta^{r,s}$ we have to obtain more detailed representation for the functions $h_r(x)$. Using (11) and (18) we can write:

$$h_r(x)=(E+S^+)\,e^{ix_r}=e^{ix_r}+\sum_{n\in\mathscr{M}_1}C^r_n e_n(x)$$

Here we denote by $\mathscr{M}_z=\{n=(n(k)), k\in Z\mid r(n)=z\}$ the set of multi-indices with charge z, $z\in Z$; C^r_n are constants such that

$$C^r_n=R^r_n(C\sqrt{\beta})^{(1/2)\,d_{\{r,\,\mathrm{supp}\,n\}}+(1/4)\,|n|} \tag{47}$$

and

$$\sup_r\sum_n |R^r_n|<D \tag{48}$$

where D is a constant, and $d_{\{r, \operatorname{supp} n\}}$ is the length of a minimal connected set of bonds such that the set containes the point r and all points of $\operatorname{supp} n$. Then for $r \neq s$ we have

$$
\begin{aligned}
-\frac{1}{2}\Delta^{r,s} &= \sum_{k\in Z} \frac{\partial}{\partial x_k}\left(e^{ix_r} + \sum_{n\in\mathscr{M}_1} C_r^n e_n(x)\right)\cdot\frac{\partial}{\partial x_k}\left(e^{ix_s} + \sum_{m\in\mathscr{M}_1} C_m^s e_m(x)\right)\\
&= -e^{ix_r}\sum_{\substack{m\in\mathscr{M}_1:\\ r\in\operatorname{supp} m}} C_m^s m(r)\, e_m(x) - e^{ix_s}\sum_{\substack{n\in\mathscr{M}_1:\\ s\in\operatorname{supp} n}} C_n^r n(s)\, e_n(x)\\
&\quad -\sum_{k\in Z}\sum_{\substack{n,m\in\mathscr{M}_1:\\ k\in\operatorname{supp} m\cap\operatorname{supp} n}} C_n^r C_m^s n(k)\, m(k)\, e_n(x)\, e_m(x)\\
&= \sum_{w\in\mathscr{M}_2} d_w^{r,s} e_w(x)
\end{aligned}
$$

By estimates (47) and (48) we have for every $w \in \mathscr{M}_2$:

$$
\begin{aligned}
|d_w^{r,s}| &\leqslant \sum_{\substack{n,m\in\mathscr{M}_1:\, w=n\cup m,\\ \operatorname{supp} n\cap\operatorname{supp} m\neq\varnothing}} \sum_{k\in\operatorname{supp} n\cap\operatorname{supp} m} |C_n^r|\cdot|C_m^s|\cdot|n(k)|\cdot|m(k)|\\
&\leqslant K_1 \sum_{\substack{n,m\in\mathscr{M}_1:\, w=n\cup m,\\ \operatorname{supp} n\cap\operatorname{supp} m\neq\varnothing}} (\alpha_2\sqrt{\beta})^{(1/2)\, d_{\{r,\operatorname{supp} n\}} + (1/2)\, d_{\{s,\operatorname{supp} m\}} + (1/4)\,|n| + (1/4)\,|m|}\\
&\qquad |\operatorname{supp} n|\cdot|R_n^r|\cdot|R_m^s|\cdot\left(\frac{\alpha_1}{\alpha_2}\right)^{(1/2)\, d_{\{r,\operatorname{supp} n\}} + (1/2)\, d_{\{s,\operatorname{supp} m\}} + (1/4)\,|n| + (1/4)\,|m|}\\
&\leqslant K_1(\alpha_2\sqrt{\beta})^{(1/2)\, d_{\{r,s,\operatorname{supp} w\}} + (1/4)\,|w|}\\
&\qquad \times\left(\sum_{n\in\mathscr{M}_1} |\operatorname{supp} n|\cdot|R_n^r|\cdot\left(\frac{\alpha_1}{\alpha_2}\right)^{(1/2)\, d_{\{r,\operatorname{supp} n\}} + (1/4)\,|n|}\right)\\
&\qquad \times\left(\sum_{m\in\mathscr{M}_1} |R_m^s|\cdot\left(\frac{\alpha_1}{\alpha_2}\right)^{(1/2)\, d_{\{s,\operatorname{supp} m\}} + (1/4)\,|m|}\right)\\
&\leqslant K_2(\alpha_2\sqrt{\beta})^{(1/2)\, d_{\{r,s,\operatorname{supp} w\}} + (1/4)\,|w|}
\end{aligned}
$$

Here K_1, K_2, $C < \alpha_1 < \alpha_2$ are constants.

Lemma 8 is proved.

A.2. The Proof of Lemma 9

The proof is based on the estimate (18). We denote by $\mathcal{Q}$ the space of bounded operators from L_2 to $L_{>2}$ with norm

$$|||Q||| = \sup_{r \neq s} \sum_w |Q_w^{r,s}| \left(\frac{1}{b\sqrt{\beta}}\right)^{(1/2)\, d_{\{r,s,\mathrm{supp}\, w\}} + (1/4)\,|w|}$$

where $Q: L_2 \to L_{>2}$, $b > \alpha_2$ is a constant, and α_2 is the same constant as in the estimates (26), (28), (29). The equation (23) on the operator S_2 implies that

$$S_2 = -H_{11}^{-1} H_{10} + H_{11}^{-1} S_2 H_{00} + H_{11}^{-1} S_2 H_{01} S_2 \equiv \mathcal{F} S_2 \tag{49}$$

where $\mathcal{F} : \mathcal{Q} \to \mathcal{Q}$ is a mapping in the space $\mathcal{Q}$.

We shall prove that the mapping $\mathcal{F}$ is a contraction on a ball $\mathcal{B}_q \subset \mathcal{Q}$:

$$\mathcal{B}_q = \{ Q \in \mathcal{Q} : |||Q||| < q \}$$

To do this we have to get estimates for every terms in the equation (49). From (29) we obtain

$$|(H_{10})_w^{r,s}| < d_w^{r,s} (b\sqrt{\beta})^{(1/2)\, d_{\{r,s,\mathrm{supp}\, w\}} + (1/4)\,|w|}$$

with $b > \alpha_2$, and

$$\sup_{r \neq s} \sum_w d_w^{r,s} < \tilde{D}$$

Hence

$$|||H_{11}^{-1} H_{10}||| \leqslant \tfrac{1}{4}(1 + k_1 \beta)\, |||H_{10}||| \leqslant \tfrac{1}{4}(1 + k_1 \beta)\, \tilde{D}$$

Then using the representation

$$(S_2 H_{00})(h_r h_s)$$
$$= \sum_{r'} m(r - r')\, S_2(h_{r'} h_s) + \sum_{s'} m(s - s')\, S_2(h_r h_{s'}) + \sum_{r' \neq s'} (g_1)_{r',s'}^{r,s}\, S_2(h_{r'} h_{s'})$$

where the function $g_1 = g_1^{r,s}$ is defined in the Corollary to Lemma 8, we have

$$
\begin{aligned}
|||S_2 H_{00}||| &= \sup_{r \neq s} \sum_w |(S_2 H_{00})_w^{r,s}| \left(\frac{1}{b\sqrt{\beta}}\right)^{(1/2)\, d_{\{r,s,\mathrm{supp}\, w\}} + (1/4)\,|w|} \\
&\leqslant \left(\sup_r \sum_{r'} \frac{|m(r-r')|}{(b\sqrt{\beta})^{(1/2)\,|r-r'|}}\right) \sup_{\substack{r', s:\\ r' \neq s}} \sum_w |(S_2)_w^{r',s}| \\
&\quad \times \left(\frac{1}{b\sqrt{\beta}}\right)^{(1/2)\, d_{\{r',s,\mathrm{supp}\, w\}} + (1/4)\,|w|} \\
&\quad + \left(\sup_s \sum_{s'} \frac{|m(s-s')|}{(b\sqrt{\beta})^{(1/2)\,|s-s'|}}\right) \sup_{\substack{s', r:\\ s' \neq r}} \sum_w |(S_2)_w^{r,s'}| \\
&\quad \times \left(\frac{1}{b\sqrt{\beta}}\right)^{(1/2)\, d_{\{r,s',\mathrm{supp}\, w\}} + (1/4)\,|w|} \\
&\quad + \left(\sup_{r \neq s} \sum_{r' \neq s'} \frac{|(g_1)_{r',s'}^{r,s}|}{(b\sqrt{\beta})^{(1/2)\,|r-r'| + (1/2)\,|s-s'|}}\right) \\
&\quad \times \sup_{\substack{r', s':\\ r' \neq s'}} \sum_w |(S_2)_w^{r',s'}| \left(\frac{1}{b\sqrt{\beta}}\right)^{(1/2)\, d_{\{r',s',\mathrm{supp}\, w\}} + (1/4)\,|w|} \\
&\leqslant (2 + C_1\sqrt{\beta})\; |||S_2||
\end{aligned}
$$

Thus

$$
|||H_{11}^{-1} S_2 H_{00}||| \leqslant \tfrac{1}{4}(1 + k_1\beta)(2 + k_2\sqrt{\beta})\; |||S_2|||
$$

Similar reasoning shows that

$$
|||H_{11}^{-1} S_2 H_{01} S_2||| \leqslant \tfrac{1}{4}(1 + k_1\beta)\, k_3\beta\; |||S_2|||^2
$$

Here $C_1, k_j, j = 1, 2, 3$ are constants.

From the above estimates it is easy to see that for small enough β the mapping $\mathscr{F}$ is a contraction on a ball $\mathscr{B}_q$, where $\frac{1}{2}\tilde{D} < q < (\frac{1}{2} + \varepsilon)\,\tilde{D}$, $\varepsilon = \varepsilon(\beta)$ is small. Hence there exists the unique solution S_2 of the equation (49) with $|||S_2||| < q$.

Lemma 9 is proved.

A.3. The Proof of the Proposition

We shall use in our proof the expression (33) for the kernel $K(r, s, r', s')$.

1. First we consider the case when $|r-s|=1$, and suppose that $r<s$. Using (46) we shall compute the function

$$\Delta^{r,s}(x)=g_1^{r,s}(x)+g_2^{r,s}(x), \qquad r\neq s$$

and then we shall separate the function $g_1^{r,s}(x)\in L_2$ from $g_2^{r,s}(x)\in L_{>2}$. For this purpose we need a detailed representation for the basis vectors $h_r(x)$ in the one-particle invariant subspace.

Using the equation (8) for the operator S^+ we can decompose S^+ in the series

$$S^+=-\sum_{k=0}^{\infty}(H_{11}^{(1)})^{-(k+1)}H_{10}^{(1)}(H_{00}^{(1)})^k+R \tag{50}$$

with $\|R\|=O(\beta^3)$. From (11) and (50) we get the following representation for $h_r(x)$, $r\in Z$:

$$\begin{aligned}
h_r=&\exp\{ix_r\}-\frac{\beta}{8}\sum_{r':|r-r'|=1}\exp\{2ix_r-ix_{r'}\}\\
&+a_1\beta^2\sum_{\substack{r':|r-r'|=1\\ r'':|r''-r'|=1}}\exp\{2ix_{r'}-ix_{r''}\}\\
&+a_2\beta^2\sum_{r':|r-r'|=1}\exp\{3ix_r-2ix_{r'}\}+a_3\beta^2\exp\{3ix_r-ix_{r'}-ix_{r*}\}\\
&+a_4\beta^2\sum_{\substack{r',r^*:\\ |r-r'|=|r-r^*|=1}}\exp\{ix_r+ix_{r'}-ix_{r*}\}\\
&+a_5\beta^2\sum_{r'':|r-r''|=2}\exp\{2ix_r-ix_{r''}\}\\
&+a_6\beta^2\sum_{\substack{r':|r-r'|=1\\ r'':|r''-r'|=1,\,r''\neq r}}\exp\{2ix_r-2ix_{r'}+ix_{r''}\}+O(\beta^3)
\end{aligned}$$

Here a_j, $j=1,\dots,6$ are constants, $a_4=-1/8$; $|r-r'|=|r-r^*|=1$. The analogous representation is valid for the basis vectors $h_r^{(-)}=\overline{h_r}$ in the space $\mathscr{H}_1^-$.

If we insert this expression in (46) we obtain

$$\begin{aligned}\Delta^{r,s}(x) &= -2\left(\frac{\partial h_r(x)}{\partial x_r}\cdot\frac{\partial h_s(x)}{\partial x_r}+\frac{\partial h_r(x)}{\partial x_s}\cdot\frac{\partial h_s(x)}{\partial x_s}\right)+O(\beta^3)\\ &=\frac{\beta}{4}(\exp\{2ix_s\}+\exp\{2ix_r\})+\frac{\beta^2}{4}(\exp\{ix_s+ix_{s'}\}+\exp\{ix_r+ix_{r'}\})\\ &\quad-\frac{\beta^2}{8}\exp\{ix_s+ix_r\}+O(\beta^2,n)+O(\beta^3)\end{aligned}$$

where we denote by $O(\beta^2, n)$ a linear combination of the vectors $e_n(x)$ of the form:

$$\sum_n c_n e_n(x), \qquad \text{where} \quad |c_n|\leqslant C\beta^2, \qquad \text{and} \qquad \tfrac{1}{2}|\text{supp}\, n|+\tfrac{1}{4}|n|\geqslant\tfrac{3}{2}$$

Thus

$$g_1^{r,s}(x)=\frac{\beta^2}{4}(h_s\cdot h_{s'}+h_r\cdot h_{r'})-\frac{\beta^2}{8}h_s\cdot h_r+O(\beta^3)\in L_2$$

$$\tilde{g}_2^{r,s}(x)=-\frac{\beta}{4}(h_s\cdot h_s+h_r\cdot h_r)+O(\beta^2,n)+O(\beta^3)\in L_{>2}$$

with $r'=r-1<r<s<s'=s+1$.

Using the expression for the operator S_2 which is similar to the series (50) we have

$$S_2(h_r\cdot h_s)=-\sum_{k=0}^{\infty}H_{11}^{-(k+1)}H_{10}H_{00}^k(h_r\cdot h_s)+O(\beta^3)=-\tfrac{1}{2}\,\tilde{g}_2^{r,s}(x)+O(\beta^2)$$

Therefore

$$\begin{aligned}&H_{01}S_2(h_r\cdot h_s)\\ &\quad=-\frac{1}{2}H_{01}\tilde{g}_2^{r,s}+O(\beta^3)=-\frac{\beta^2}{8}(h_s\cdot h_{s'}+h_r\cdot h_{r'}+2h_s\cdot h_r)+O(\beta^3)\end{aligned}$$

with $r'=r-1<r<s<s'=s+1$. Finally by (33) we get the representations (39) and (41).

The function $K^0(r,s;r',s')$ can be investigated in a similar way.

2. The estimate (34) implies that the study of the functions $K(r, s; r', s')$ and $K^0(r, s; r', s')$, when $|r-s| \leqslant 10$, suffices to prove (42). However repeating previous reasoning in the case, when $|r-s| \geqslant 2$, we obtain that estimate (42) is valid for any r, s such that $|r-s| \geqslant 2$.

The proposition is proved.

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